Research Article

Gwanghyun Jo and Do Young Kwak*

A Semi-Uniform Multigrid Algorithm for Solving Elliptic Interface Problems

https://doi.org/10.1515/cmam-2020-0039
Received March 23, 2020; revised July 14, 2020; accepted August 27, 2020

Abstract: We introduce a new geometric multigrid algorithm to solve elliptic interface problems. First we discretize the problems by the usual P_1-conforming finite element methods on a semi-uniform grid which is obtained by refining a uniform grid. To solve the algebraic system, we adopt subspace correction methods for which we use uniform grids as the auxiliary spaces. To enhance the efficiency of the algorithms, we define a new transfer operator between a uniform grid and a semi-uniform grid so that the transferred functions satisfy the flux continuity along the interface. In the auxiliary space, the system is solved by the usual multigrid algorithm with a similarly modified prolongation operator. We show $\mathcal{W}$-cycle convergence for the proposed multigrid algorithm. We demonstrate the performance of our multigrid algorithm for problems having various ratios of parameters. We observe that the computational complexity of our algorithms are robust for all problems we tested.

Keywords: Geometric Multigrid, Elliptic Interface Problem, Semi-Uniform Grid, $\mathcal{W}$-Cycle Convergence

MSC 2010: 65N12, 65N30, 65N55

1 Introduction

Multigrid (MG) algorithm is one of the most efficient solvers for linear systems arising from the discretization of partial differential equations (PDEs) [5, 13, 24]. However, it is hard to develop robust geometric MG algorithms when there exist some interfaces in the domain across which the coefficients are distinct. This is because the PDE needs to be discretized on the fitted grids, which results in the non-uniformity of the matrix structure. Thus one often uses algebraic multigrid methods [12, 26, 27] or various PCG type of solvers. However, efficient fast solvers are rarely available. For finite difference type MG algorithms for interface problems were considered in [1, 2].

Recently, the authors developed a robust geometric MG algorithm for interface problems [17, 18] discretized using immersed finite element method [19, 20, 23]. The idea in [17] is to use uniform grids for the interface problems where the basis functions are modified instead. Since the discretized system was constructed on uniform grids, it is possible to develop geometric multigrid algorithms for problems having the interface problems.

In this work, we develop a new MG algorithm for interface problems through some other approach. We use the usual P_1-conforming finite element method on a semi-uniform grid. Our semi-uniform grids are obtained from the uniform grid by subdividing the interface element into three triangles using the intersection points with the interface. To solve the discretized system, we adopt subspace correction ideas in [13, 21] by choosing the uniform grids space as an auxiliary space. First, we consider a two-grid method using a uniform grid

---

***Corresponding author: Do Young Kwak,** Department of Mathematical Sciences, KAIST, Daejeon, Republic of Korea, e-mail: kdy@kaist.ac.kr. https://orcid.org/0000-0002-5743-1501
Gwanghyun Jo, Department of Mathematics, Kunsan National University, Gunsan-si, Jeollabuk-do, Republic of Korea, e-mail: gwanghyun@kunsan.ac.kr. https://orcid.org/0000-0002-0635-2897

as an auxiliary subgrid. In this step, the values only at the interface points are interpolated. For the inner algorithm, the modified prolongation operator is used on uniform grids. We will call our method a semi-uniform multigrid (SUMG) algorithm.

The efficiency of the subspace correction methods relies on the transfer operator which updates corrections of values on uniform grids onto the semi-uniform grid. We define the transfer operator so that the transferred functions satisfy the local flux conditions along the interface. For the systems on the uniform grids, we use similar ideas for the prolongation operators in the multigrid algorithm on auxiliary spaces. In this way, we are able to develop a robust algorithm.

We analyze our multigrid algorithm using the frameworks of [7] where the analysis of multigrid algorithms on non-nested or non-inherited spaces is provided. In our algorithm, all subspaces are nested. However, the bilinear forms are not inherited between grids.

We consider the following elliptic interface problem on a convex polygonal (polyhedral) domain Ω in $\mathbb{R}^n$ ($n = 2, 3$):

$$-\nabla \cdot \beta \nabla u = f \quad \text{in } \Omega, \tag{1.1a}$$

$$[u]_\Gamma = 0, \tag{1.1b}$$

$$\left[\beta \frac{\partial u}{\partial \mathbf{n}}\right]_\Gamma = 0, \tag{1.1c}$$

$$u = 0 \quad \text{on } \partial\Omega, \tag{1.1d}$$

where $f \in L^2(\Omega)$, and $\Gamma \subset \Omega$ is an interface which divides the domain into two subdomains Ω^+ and Ω^-. Here, $[\,\cdot\,]_\Gamma$ implies the jumps of functions along Γ, i.e.,

$$[u]_\Gamma = u|_{\Omega^-} - u|_{\Omega^+},$$

$$\left[\beta \frac{\partial u}{\partial \mathbf{n}}\right]_\Gamma = \beta|_{\Omega^-} \frac{\partial u|_{\Omega^-}}{\partial \mathbf{n}} - \beta|_{\Omega^+} \frac{\partial u|_{\Omega^+}}{\partial \mathbf{n}},$$

where $\mathbf{n}$ is an outward normal vector to Ω^-. We assume that Γ is a C^1-curve. The coefficient β is discontinuous across the interface Γ, where $\beta = \beta^+ \in C(\Omega^+)$ and $\beta = \beta^- \in C(\Omega^-)$.

We introduce some function spaces and notations. For any bounded domain D and positive integer m, let $H^m(D)$ be the usual Sobolev space of order m with the norm denoted by $\|\cdot\|_{m,D}$. We define $H_0^1(D)$ as a set of functions in $H^1(D)$ vanishing on ∂D. We denote the dual space of $H_0^1(D)$ by $H^{-1}(D)$. For any real number between integer m and $m+1$, we define fractional Sobolev space $H^s(D)$ as the interpolation between $H^m(D)$ and $H^{m+1}(D)$. We need to define subspaces of $H_0^1(\Omega)$ and $H^{1+\alpha}(\Omega)$, which satisfy the interface conditions

$$H_{0,\Gamma}^1(\Omega) := \{u \in H_0^1(\Omega) : [u]_\Gamma = [\beta \nabla u \cdot \mathbf{n}] = 0\},$$

$$H_{0,\Gamma}^{1+\alpha}(\Omega) := H_{0,\Gamma}^1(\Omega) \cap H^{1+\alpha}(\Omega).$$

Integration by parts gives the variational problem for the model problem (1.1): find $u \in H_0^1(\Omega)$ such that

$$\int_{\Omega^-} \beta \nabla u \cdot \nabla v \, dx + \int_{\Omega^+} \beta \nabla u \cdot \nabla v \, dx = \int_{\Omega} f v \, dx \tag{1.2}$$

for all $v \in H_0^1(\Omega)$.

We state the regularity theorem regarding the solutions of the elliptic interface problems [3, 5, 16, 25].

Proposition 1.1. *There exists an* $0 < \alpha \le 1$, *and a unique solution* $u \in H^2(\Omega^-) \cap H^2(\Omega^+) \cap H_{0,\Gamma}^{1+\alpha}(\Omega)$ *of problem* (1.2) *which satisfies*

$$\|u\|_{H^{1+\alpha}(\Omega)} \le C(\beta) \|f\|_{H^{-1+\alpha}(\Omega)},$$

where $C(\beta)$ *is some positive constant depending on* β.

For the simplicity of the presentation, we assume $n = 2$, even though the case $n = 3$ can be similarly treated. The rest of the paper is organized as follows. In Section 2, we review some results on the P_1-conforming Galerkin methods for elliptic interface problems where unfitted grids are used. We propose our semi-uniform multigrid algorithm in Section 3 and we prove the contracting properties of the multigrid algorithm in Section 4. The numerical results are given in Section 5 and conclusion follows in Section 6.

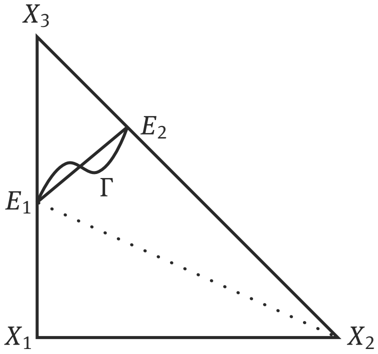

Figure 1: An interface element.

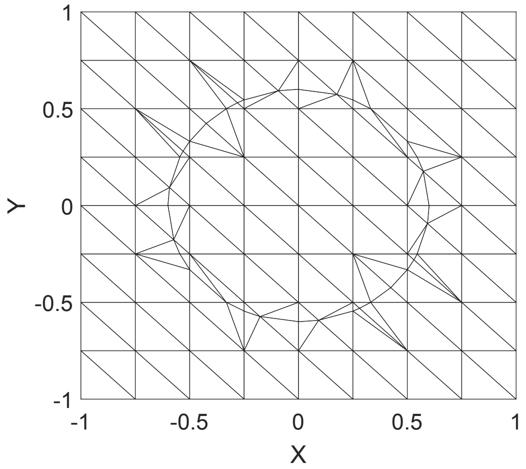

Figure 2: Refined grid $\mathcal{F}_h$ from uniform grids $\mathcal{T}_h$ with $h = 0.25$ where interface is given by circle $x^2 + y^2 = 0.6^2$.

2 P_1-Conforming Finite Element Method on Semi-Uniform Grids

We define a semi-uniform grid by sequential steps. First, we let $\mathcal{T}_h$ be a uniform triangulation of Ω by right triangles with size h. We call $T \in \mathcal{T}_h$ is an interface element if T is cut by the interface. Otherwise, T is non-interface element.

We define a mesh $\mathcal{F}_h$ which is a refinement of $\mathcal{T}_h$. We assume that elements are cut by interface at no more than two points. This assumption is reasonable if we choose h sufficiently small, see [10]. Firstly, the non-interface elements in $\mathcal{F}_h$ are inherited from $\mathcal{T}_h$. If $T \in \mathcal{T}_h$ is an interface element, then T is divided into two or three triangles by including two interface points as new nodes. For example, suppose T is cut by interface at two edges with intersection points E_1 and E_2 resulting in a triangle and a quadrilateral region (see Figure 1). Then quadrilateral region is divided into two sub triangles connecting two nodes in such a way the resulting triangle satisfy the maximum angle condition [9]. Figure 2 shows the example of $\mathcal{F}_h$ when the interface has circular shape.

We consider two discretizations for problem (1.1).

2.1 P_1-Conforming Methods on the Fitted Grid

We describe the usual P_1-conforming Galerkin methods on $\mathcal{F}_h$. Let T be a triangle in $\mathcal{F}_h$. Let $S_h(T)$ be the set of linear polynomials on T. Let $S_h(\Omega)$ be the usual continuous, piecewise linear finite element (FE) space based on $\mathcal{F}_h$ satisfying homogeneous boundary condition. We associate bilinear form

$$a_h(u,v) := \sum_{T\in\mathcal{F}_h}\Bigg(\int_{T\cap\Omega^-}\beta^-\nabla u\cdot\nabla v + \int_{T\cap\Omega^+}\beta^+\nabla u\cdot\nabla v\Bigg), \quad u,v\in H^1(\Omega).$$

We define a Galerkin method as usual: find $\tilde{u}_h \in S_h(\Omega)$ satisfying

$$a_h(\tilde{u}_h, v_h) = (f, v_h) \quad \text{for all } v_h \in S_h(\Omega), \tag{2.1}$$

where $(\cdot,\cdot)$ is the usual L^2-inner product. The following convergence theorem is proven in [4].

Theorem 2.1. *There exists a unique solution for* (2.1). *Suppose u is solution of* (1.1) *and let $\tilde{u}_h$ be a solution of* (2.1). *Then the following holds:*

$$\|u-\tilde{u}_h\|_{L^2(\Omega)} + h\|\nabla(u-\tilde{u}_h)\|_{L^2(\Omega)} \le Ch^2\|f\|_{L^2(\Omega)}.$$

2.2 P_1-Conforming Methods on Uniform Grid

Let $V_h(\Omega)$ be the set of continuous, piecewise linear polynomials on the triangulation $\mathcal{T}_h$. Let us recall the property of the bilinear form $a_h(\cdot,\cdot)$. We define L^2 and H^1-norms on $H^1(\Omega)$ as

$$\|u\|_{m,h} := \sum_{T\in\mathcal{T}_h} \|u\|_{m,T}, \quad u \in H^1(\Omega),$$

where $m = 0$ or 1. It is well known that the energy-like norm on $H^1(\Omega)$, defined by $|||u|||_h := \sqrt{a_h(u,u)}$, is equivalent to H^1-norm, i.e., there exists some $C > 0$ such that

$$\frac{1}{C}\|u\|_{1,h} \le |||u|||_h \le C\|u\|_{1,h}. \tag{2.2}$$

The following theorem regarding the inverse inequality is also well known [11].

Theorem 2.2. *There exists a constant $C > 0$ such that for all ϕ in $S_h(\Omega)$ following holds:*

$$a_h(\phi,\phi) \le Ch^{-2}\|\phi\|^2_{L^2(\Omega)}. \tag{2.3}$$

Let $\pi_h : H^{1+\alpha}(\Omega) \to V_h(\Omega)$ be the interpolation operator defined by

$$(\pi_h u)(X) = u(X) \quad \text{for all nodes } X \text{ of } \mathcal{T}_h.$$

For three dimension case, we assume $\alpha > 0.5$ so that the interpolation operator on $\mathcal{T}_h$ can be well defined. The following results are well known [11].

Theorem 2.3. *There exist $C > 0$ such that, for all $w \in H^{1+\alpha}(\Omega)$ following holds:*

$$\|w - \pi_h w\|_{m,h} \le Ch^{1-m+\alpha}\|w\|_{H^{1+\alpha}(\Omega)}, \quad m = 0, 1, \tag{2.4}$$

$$\|\pi_h w\|_{m,h} \le C\|w\|_{m,h}, \quad m = 0, 1. \tag{2.5}$$

Now, P_1-conforming Galerkin methods on uniform grids reads: Find $u_h \in V_h$ such that

$$a_h(u_h,\phi) = (f,\phi) \tag{2.6}$$

for all $\phi \in V_h$. The following convergence theorem is a result of Theorem 2.3 and Céa's Lemma.

Theorem 2.4. *Suppose u is solution of* (1.1) *and u_h be solution of* (2.6). *Then following holds:*

$$\|u - u_h\|_{L^2(\Omega)} + h^\alpha\|u - u_h\|_{1,h} \le C(\beta)h^{2\alpha}\|w\|_{H^{1+\alpha}(\Omega)}.$$

3 Multigrid Algorithm

Using auxiliary space preconditioning method [14, 15, 28] as a preconditioner for conjugate gradient method is an efficient strategy to solve algebraic systems. We adopt the idea of auxiliary space preconditioning methods in this work. For the simplicity of presentation, we assume Ω is a rectangular region. We develop a new type of multigrid algorithm for $S_h(\Omega)$, where the space $V_h(\Omega)$ is used as an auxiliary space. We define sequential triangulations $\mathcal{T}_{h_k}$ with size $h_k = h_0 \cdot 2^{-k}$, $k = 1,\dots,J$, for Ω. As in the previous section, semi-uniform triangulation $\mathcal{F}_{h_J}$ is a fitted grid obtained from the finest grids $\mathcal{T}_{h_J}$. We note the inclusion relationships between the subspaces

$$V_{h_1} \subset V_{h_2} \subset \cdots \subset V_{h_J} \subset S_{h_J}.$$

From now on, we replace the subscript h_k by the subscript k when there is no confusion. For example,

$$\mathcal{T}_k = \mathcal{T}_{h_k}, \quad a_k(\cdot,\cdot) = a_{h_k}(\cdot,\cdot), \quad S_k(\Omega) = S_{h_k}(\Omega), \quad V_k(\Omega) = V_{h_k}(\Omega).$$

We introduce some notations. We define $\widetilde{A}_J : S_J(\Omega) \to S_J(\Omega)$ so that

$$a_J(\phi,\psi) = (\widetilde{A}_J\phi,\psi)$$

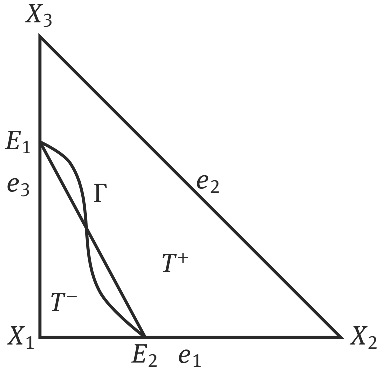

Figure 3: An interface element.

holds for all ϕ, ψ in $S_J(\Omega)$. Similarly, we define $A_k(\Omega) : V_k(\Omega) \to V_k(\Omega)$, $k = 1, 2, \dots, J$, so that

$$a_k(\phi, \psi) = (A_k\phi, \psi)$$

holds for all ϕ, ψ in $V_k(\Omega)$.

We define an operator $\widehat{\gamma}_k : V_k(\Omega) \to L^2(\Omega)$, which will play an important role in our subspace correction multigrid algorithm. Suppose T is a non-interface element in $\mathcal{T}_k$. Then $\widehat{\gamma}_k(\phi)|_T = \phi|_T$. Suppose T is an interface element (see Figure 3). We define $\widehat{\gamma}_k(\phi)|_T$ as a piecewise linear function on T:

$$\widehat{\gamma}_k(\phi)|_T = \begin{cases} a^+ + b^+x + c^+y, & (x,y) \in T^+, \\ a^- + b^-x + c^-y, & (x,y) \in T^-, \end{cases} \tag{3.1}$$

where the coefficients in (3.1) are determined by nodal values and interface conditions (1.1b) and (1.1c) as

$$\begin{aligned} \widehat{\gamma}_k(\phi)|_T(X_i) &= \phi_{k-1}(X_i), && i = 1, 2, 3, \\ \widehat{\gamma}_k(\phi)|_{T^+}(E_i) &= \widehat{\gamma}_k(\phi)|_{T^-}(E_i), && i = 1, 2, \\ \int\limits_{\overline{E_1E_2}} \beta^+\nabla\widehat{\gamma}_k(\phi)|_{T^+} \cdot \mathbf{n}_{\Gamma_h} &= \int\limits_{\overline{E_1E_2}} \beta^-\nabla\widehat{\gamma}_k(\phi)|_{T^-} \cdot \mathbf{n}_{\Gamma_h}. \end{aligned}$$

The following result is given in [20, 22].

Theorem 3.1. *There exists a constant $C > 0$ such that for all $w \in H^2(\Omega^-) \cap H^2(\Omega^+) \cap H^{1+\alpha}_{0,\Gamma}(\Omega)$, the following holds:*

$$\|w - \widehat{\gamma}_k(\pi_k w)\|_{m,h_k} \le C h_k^{1-m+\alpha} \|w\|_{H^{1+\alpha}(\Omega)}, \quad m = 0, 1. \tag{3.2}$$

3.1 Semi-Uniform Multigrid Algorithm

We now explain our multigrid algorithm using the auxiliary space $V_J(\Omega)$. We write system (2.1) as the matrix equation

$$\widetilde{A}_J x = \widetilde{f}_J.$$

First, we let $\widetilde{R}_J$ be the smoothing operator (say Jacobi or Gauss–Seidel operator) for $\widetilde{A}_J$ and let $\widetilde{R}_J^t$ be the transpose of $\widetilde{R}_J$. To use the subspace correction idea, we need a transfer operator $\mathcal{Q}_U^F : \phi \in V_J(\Omega) \to S_J(\Omega)$. It suffices to define the nodal values of $\mathcal{Q}_U^F\phi$ at all nodes. First, we define

$$\mathcal{Q}_U^F\phi(A) = \phi(A)$$

when A is a node of the $\mathcal{T}_J$ (for example $A = X_i$ ($i = 1, 2, 3, 4$) in Figure 4). Suppose A is a node of $\mathcal{F}_J$ but not a node of $\mathcal{T}_J$ (for example $A = E$ in Figure 4). Then we define

$$\mathcal{Q}_U^F\phi(A) := \frac{1}{2}(\widehat{\gamma}_J(\phi)|_{T_1}(A) + \widehat{\gamma}_J(\phi)|_{T_2}(A)),$$

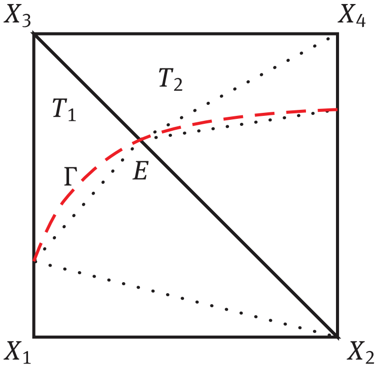

Figure 4: T_1 and T_2 are adjacent elements in $\mathcal{T}_J$, and sub-triangles having a dotted edge are elements in $\mathcal{F}_J$.

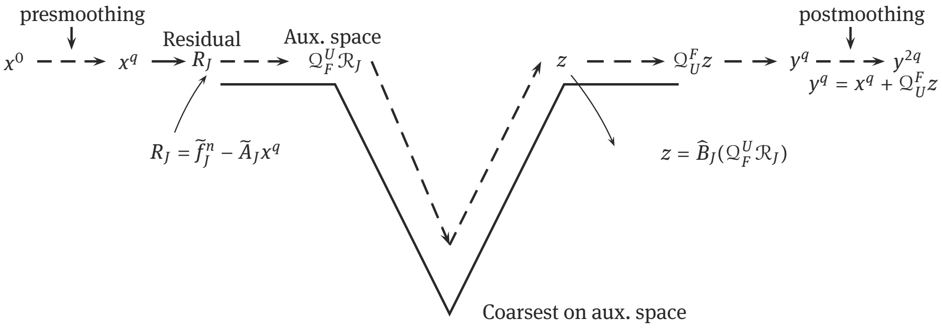

Figure 5: One cycle of **SUMG**$_J$.

where T_1 and T_2 are adjacent elements in $\mathcal{T}_J$ which share A as a common node of $\mathcal{F}_J$. We define the operator $\mathcal{Q}_F^U$ from $S_J(\Omega)$ to $V_J(\Omega)$ as the transpose of $\mathcal{Q}_U^F$. We assume a sequence of (symmetric) inner grid multigrid operators $\widehat{B}_k : V_k(\Omega) \to V_k(\Omega)$, $k = 1, \dots, J$, are defined (see next subsection). We propose a subspace correction multigrid algorithm:

Algorithm SUMG$_J$. Proceed as follows.

(1) Set $x^0 = 0$ and $z^0 = 0$.

(2) Define x^i for $i = 1, \dots, q$ by

$$x^i = x^{i-1} + \tilde{R}_k(\tilde{f}_J - \tilde{A}_J x^{i-1}).$$

(3) Restrict the residual to $V_J(\Omega)$: $\mathcal{Q}_F^U(\tilde{f}_J^n - \tilde{A}_J x^q)$.

(4) Define $z = \widehat{B}_J \mathcal{Q}_F^U(\tilde{f}_J^n - \tilde{A}_J x^q)$.

(5) Define y^q by $y^q = x^q + \mathcal{Q}_U^F z$.

(6) Define y^i for $i = q + 1, \dots, 2q$ by

$$y^i = y^{i-1} + \tilde{R}_J^t(\tilde{f}_J^n - \tilde{A}_J^n y^{i-1}).$$

(7) Set $\mathbf{SUMG}_J \tilde{f}_J^n = y^{2q}$.

Here, q is the number of pre and post-smoothings (see Figure 5). We note that **SUMG**$_J$ is a symmetric operator. Figure 5 illustrates one cycle of semi-uniform multigrid algorithm.

Now we describe the $\widehat{B}_k$ operator in the following subsection.

3.2 A Multigrid Algorithm $\widehat{B}_k$ on Uniform Grids

We describe the inner multigrid algorithm $\widehat{B}_k$ ($k = 1, \dots, J$) to solve the system of the form

$$A_J u_J = g_J.$$

We first define the prolongation operator $\widehat{I}_k : V_{k-1}(\Omega) \to V_k(\Omega)$.

$$\widehat{I}_k v(X) = \begin{cases} v(X) & \text{if } X \text{ is a node of } \mathcal{T}_{k-1}, \\ \frac{1}{2}(\widehat{\gamma}_{k-1}(\phi)|_{T_1}(X) + \widehat{\gamma}_{k-1}(\phi)|_{T_2}(X)) & \text{if } X \text{ is a midpoint of an edge } e \text{ shared} \\ & \text{by two triangles } T_1, T_2 \in \mathcal{T}_{k-1}. \end{cases}$$

The restriction operator P^0_{k-1} is defined as the adjoint operators of $\widehat{I}_k$ with respect to L^2-inner product $(\cdot\,,\cdot)$, i.e., for $u \in V_k(\Omega)$ and $\phi \in V_{k-1}(\Omega)$,

$$(P^0_{k-1}u, \phi) = (u, \widehat{I}_k\phi).$$

We let R_k be a smoothing operator for A_k (say Jacobi or Gauss–Seidel operator). Now we state the inner multigrid algorithm $\widehat{B}_k$ below.

Algorithm $\widehat{B}_k$. Set $\widehat{B}_0 g_0 = A_0^{-1} g_0$. Suppose $\widehat{B}_{k-1}$ is defined. We define $\widehat{B}_k g_k$ for $g_k \in V_k(\Omega)$ in a recursive way.

(1) Set $x^0 = 0$ and $z^0 = 0$.

(2) Define x^i for $i = 1, \dots, m$ by

$$x^i = x^{i-1} + R_k(g_k - A_k x^{i-1}).$$

(3) Define y^m by $y^m = x^m + \widehat{I}_k z^p$ where z^j for $j = 1, \dots, p$ is defined by

$$z^j = z^{j-1} + \widehat{B}_{k-1}[P^0_{k-1}(g_k - A_k x^m) - A_{k-1} z^{j-1}].$$

(4) Define y^i for $i = m+1, \dots, 2m$ by

$$y^i = y^{i-1} + R^t_k(g_k - A_k y^{i-1}).$$

(5) Set $\widehat{B}_k g_k = y^{2m}$.

Note that this is almost the same as the standard MG algorithm, but the new prolongation operator $\widehat{I}_k$ is used. The case of $p = 1$ and $p = 2$ corresponds to $\mathcal{V}$ and $\mathcal{W}$-cycles respectively. We will use notation $\mathcal{V}(m, m)$ (resp. $\mathcal{W}(m, m)$) for $\widehat{B}_k$ when $p = 1$ (resp. $p = 2$).

Let us recall the semi-uniform MG algorithm in Section 3.1. We will use the notation $\mathcal{V}_q(m, m)$ for $\mathbf{SUMG}_J$ when the number of smoothings in $\mathbf{SUMG}_J$ is q and $\mathcal{V}(m, m)$ is used as a inner multigrid algorithm.

4 Convergence Analysis of Multigrid

In this section, we provide an analysis for both of the multigrid algorithms. First, we analyze algorithm $\widehat{B}_k$, from which we easily obtain the convergence of $\mathbf{SUMG}_J$. First we define $P_{k-1} : V_k(\Omega) \to V_{k-1}(\Omega)$ as the adjoint operator of $\widehat{I}_k$ with respect to a_k form, i.e., P_{k-1} satisfies

$$a_{k-1}(P_{k-1}u, v) = a_k(u, \widehat{I}_k v)$$

for all $u \in V_k(\Omega)$ and $v \in V_{k-1}(\Omega)$.

We shall use the framework of Bramble et. al [7], where the convergence of multigrid algorithm with general prolongation operators are provided. Note that our subspaces V_k are nested, but the prolongation operators are not a natural injection operator.

We state some assumptions.

(A.1) **Smoothing property.** There exists a constant $C_R > 0$ such that for all $u \in V_k(\Omega)$,

$$\frac{(u, u)}{\lambda_k} \le C_R(\widehat{R}_k u, u),$$

where λ_k is the maximum eigenvalue of A_k, $K_k = I - R_k A_k$, $K^*_k = I - R^t_k A_k$ and $\widehat{R}_k = (I - K^*_k K_k)A_k^{-1}$.

(A.2) There exists a constant $C^* > 0$ such that

$$A_k(\widehat{I}_k u, \widehat{I}_k u) \le C^* A_{k-1}(u, u) \quad \text{for all } u \in V_{k-1}(\Omega).$$

(A.3) **Regularity and approximation.** There exist a number $0 < \nu \leq 1$ and a constant $C > 0$ such that

$$|a_k((I - \widehat{I}_k P_{k-1})u, u)| \leq C_\alpha \Big(\frac{\|A_k u\|_0^2}{\lambda_k} \Big)^\nu a_k(u, u)^{1-\nu}$$

for all $u \in V_k(\Omega)$.

Then by the framework in [7], we can conclude the following result.

Theorem 4.1. *Suppose $p = 2$ and assumptions* (A.1), (A.2) *and* (A.3) *hold. If "m is sufficiently large", then we have*

$$|a_k((I - \widehat{B}_k A_k)u, u)| \leq \delta a_k(u, u) \quad \textit{for all } u \in V_k(\Omega),$$

where δ is some constant independent of k with the form

$$\delta = \frac{M}{M + m^\nu}$$

for some $M > 0$.

The constant M is a function of α, C_α and C_R, see [5, 7]. One can find explicit form of M in [5]. In a similar way, we have:

Theorem 4.2. *Under the same assumptions as Theorem 4.1, we have*

$$|a_J((I - \mathbf{SUMG}_J \widetilde{A}_J)u, u)| \leq \delta a_k(u, u) \quad \textit{for all } u \in S_J(\Omega).$$

Proof. This is a two grid algorithm using the grid $\mathcal{T}_J$ and $\mathcal{F}_J$, between which the transfer operators $\mathcal{Q}_F^U$ and $\mathcal{Q}_U^F$ are used instead of P_{k-1} and $\widehat{I}_k$. Thus the result follows exactly in the same way as Theorem 4.1. ☐

We now examine assumptions (A.1)–(A.3). It is clear that A_k is symmetric positive definite and sparse matrix. Thus, standard smoothing operators, such as Gauss–Seidel (GS) and Jacobi methods, satisfy (A.1), see [6]. Therefore, it suffices to verify (A.2) and (A.3). We remark that $\lambda_k = \mathcal{O}(h^{-2})$.

4.1 Approximation Properties of $\widehat{I}_k$

In this subsection, we prove some properties of $\widehat{I}_k$ that will play an important role in proving (A.1) and (A.2). We shall also need the following fact which trivially holds for the piecewise linear functions.

Lemma 4.3. *There exists a constant $C > 0$ such that*

$$\frac{h}{C} \sum_{i=1,2,3} |v(X_i)| \leq \|v\|_{L^2(T)} \leq Ch \sum_{i=1,2,3} |v(X_i)| \quad \textit{for all } v \in S_h(T), \tag{4.1}$$

where X_i, $i = 1, 2, 3$, are nodes of T.

We introduce some notations. We define a space of (discontinuous) piecewise linear FE space

$$P_{h_k}^{-1} := \{\phi \in L^2(\Omega) : \phi|_T \text{ is linear polynomial on } T \text{ for all } T \in \mathcal{T}_k\}.$$

For each $w \in H^2(\Omega^-) \cap H^2(\Omega^+) \cap H_{0,\Gamma}^{1+\alpha}(\Omega)$, we associate the function $D_k(w)$ in $P_{h_k}^{-1}$. We let

$$D_k(w)|_T = \pi_k(\widehat{\gamma}_{k-1}(\pi_{k-1} w)|_T)|_T$$

on each $T \in \mathcal{T}_{k-1}$. We remark that, $D_k(w)|_T$ is a continuous, piecewise linear function on each subtriangle in $\mathcal{T}_k$, but $D_k(w)$ is discontinuous in general. The following approximation property holds:

Lemma 4.4. *Let $w \in H^2(\Omega^-) \cap H^2(\Omega^+) \cap H_{0,\Gamma}^{1+\alpha}(\Omega)$. Then we have*

$$\|D_k(w) - \pi_k w\|_{m,T} \leq C h_k^{1-m+\alpha} \|w\|_{H^{1+\alpha}(T)}, \quad m = 0, 1, \tag{4.2}$$

for all $T \in \mathcal{T}_{k-1}$.

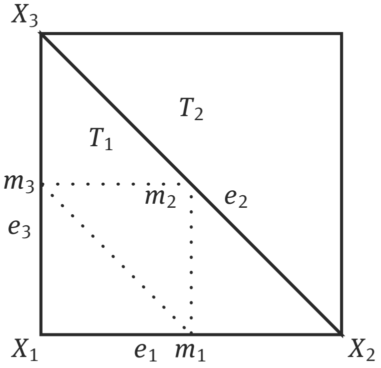

Figure 6: T_1 and T_2 are neighboring elements in $\mathcal{T}_{k-1}$ having common edge e_2.

Proof. If T is a non-interface element, this follows from the standard interpolation theory. Suppose T is an interface element. By the triangle inequality and (2.5), (2.4) and (3.2), we have

$$\begin{aligned}\|w - D_k(w)\|_{m,T} &= \|w - \pi_k(\widehat{\gamma}_{k-1}(\pi_{k-1}w)|_T)\|_{m,T}\\ &\le \|w - \pi_k w\|_{m,T} + \|\pi_k(w - \widehat{\gamma}_{k-1}(\pi_{k-1}w)|_T)\|_{m,T}\\ &\le \|w - \pi_k w\|_{m,T} + C\|w - \widehat{\gamma}_{k-1}(\pi_{k-1}w)\|_{m,T}\\ &\le Ch^{1-m+\alpha}\|w\|_{H^{1+\alpha}(T)}.\end{aligned}$$ □

Next, we study the jumps of $D_k(w)$ along the edges of $T \in \mathcal{T}_{k-1}$.

Lemma 4.5. *Let $w \in H^2(\Omega^-) \cap H^2(\Omega^+) \cap H^{1+\alpha}_{0,\Gamma}(\Omega)$. Let e be the common edge of $T_1, T_2 \in \mathcal{T}_{k-1}$. Then the following holds:*

$$\max_e |[D_k(w)]_e| \le Ch^\alpha(\|w\|_{H^{1+\alpha}(T_1)} + \|w\|_{H^{1+\alpha}(T_2)}), \tag{4.3}$$

where $[D_k(w)]_e$ is the jump of $D_k(w)$ along e.

Proof. For convenience, let $\phi_k = D_k(w) - \pi_k w$. Suppose that T_1 has nodes X_i and midpoints m_i of edge e_i $(i = 1, 2, 3)$ respectively. Without loss of generality, we assume that the common edge of T_1 and T_2 is e_2 (see Figure 6). We note that $\phi_k|_{T_1}$ is a continuous, piecewise linear function on T_1 having six degrees of freedom (at nodes X_i and mid points m_i, $i = 1, 2, 3$). By using inequality (4.1) and (4.2), we have

$$Ch_k\Big(\sum_{i=1,2,3} |\phi_k|_{T_1}(X_i)| + \sum_{i=1,2,3} |\phi_k|_{T_1}(m_i)|\Big) \le \|\phi_k\|_{0,T_1} \le Ch_k^{1+\alpha}\|w\|_{H^{1+\alpha}(T_1)}.$$

However, since $D_k(w)|_{T_1}(X_i) = \pi_k w|_{T_1}(X_i)$, we have

$$|\phi_k|_{T_1}(m_i)| \le Ch_k^\alpha\|w\|_{H^{1+\alpha}(T_1)}, \quad i = 1, 2, 3.$$

This implies

$$\max_e |(\phi_k|_{T_1})_e| \le Ch_k^\alpha\|w\|_{H^{1+\alpha}(T_1)}. \tag{4.4}$$

Similarly, we have

$$\max_e |(\phi_k|_{T_2})_e| \le Ch_k^\alpha\|w\|_{H^{1+\alpha}(T_2)}. \tag{4.5}$$

By (4.4) and (4.5), we have

$$|[\phi_k]_e|_e \le Ch_k^\alpha\big(\|w\|_{H^{1+\alpha}(T_1)} + \|w\|_{H^{1+\alpha}(T_2)}\big). \tag{4.6}$$

However, since $\pi_k w$ is continuous on Ω,

$$|[\phi_k]_e|_e = |[D_k]_e|_e.$$

Therefore, (4.6) leads to the conclusion. □

Finally, we give the main proposition regarding the approximation property of $\widehat{I}_k$.

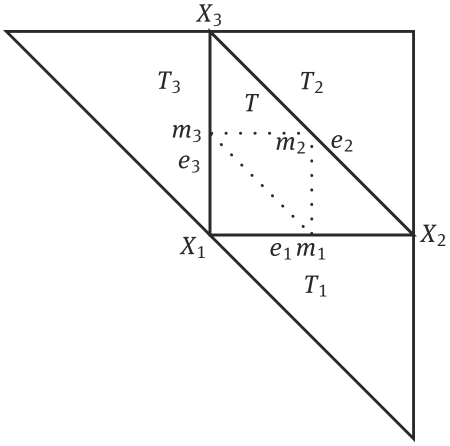

Figure 7: T is typical element in $\mathcal{T}_{k-1}$ and T_1 and T_2 and T_3 are neighboring elements of T in $\mathcal{T}_{k-1}$.

Proposition 4.6. *There exists a constant $C > 0$ such that for all $w \in H^2(\Omega^-) \cap H^2(\Omega^+) \cap H^{1+\alpha}_{0,\Gamma}(\Omega)$ and for $m = 0$ or 1, the following holds:*

$$\|\pi_k w - \widehat{I}_k \pi_{k-1} w\|_{m,h_k} \le C h_k^{1-m+\alpha} \|w\|_{H^{1+\alpha}(\Omega)}. \tag{4.7}$$

Proof. We refer to Figure 7. Suppose that T is the triangle in the center having X_1, X_2 and X_3, and midpoints of edges as m_1, m_2, and m_3. We suppose that T has neighboring elements T_i $(i = 1, 2, 3)$, where e_i is the common edge of T_i and T, respectively. By the definition of $D_k(w)$, and $\widehat{I}_{k-1}$, we have

$$\widehat{I}_{k-1}\pi_{k-1}(X_i) = D_k(w)|_T(X_i) = \pi_k w(X_i) = w(X_i), \tag{4.8}$$

$$\widehat{I}_{k-1}\pi_{k-1}(m_i) = \frac{1}{2}\big(D_k(w)|_T(m_i) + D_k(w)|_{T_i}(m_i)\big). \tag{4.9}$$

The identity $\frac{1}{2}(a + b) = a - \frac{1}{2}(a - b)$ yields

$$\widehat{I}_{k-1}\pi_{k-1}(m_i) = D_k(w)|_T(m_i) - \frac{1}{2}[D_k(w)(m_i)]_{e_i}.$$

By the triangle inequality, we have

$$|\pi_k w(m_i) - \widehat{I}_{k-1}\pi_{k-1}(m_i)| \le |\pi_k w(m_i) - D_k(w)|_T(m_i)| + \frac{1}{2}|[D_k(w)(m_i)]_{e_i}| = A_1 + A_2. \tag{4.10}$$

Using the fact that $\pi_k w$ and $D_k(w)$ are piecewise polynomials on T and equations (4.1) and (4.2), we obtain

$$A_1 \le C h_k^{-1} \|\pi_k w - D_k(w)\|_{0,T} \le C h_k^{-1} C h_k^{1+\alpha} \|w\|_{H^{1+\alpha}(T)} = C h_k^{\alpha} \|w\|_{H^{1+\alpha}(T)}. \tag{4.11}$$

By (4.3), we have

$$A_2 \le C h_k^{\alpha}\big(\|w\|_{H^{1+\alpha}(T)} + \|w\|_{H^{1+\alpha}(T_i)}\big). \tag{4.12}$$

From (4.10), (4.11) and (4.12), we obtain

$$|\pi_k w(m_i) - \widehat{I}_{k-1}\pi_{k-1}(m_i)| \le C h_k^{\alpha}\big(\|w\|_{H^{1+\alpha}(T)} + \|w\|_{H^{1+\alpha}(T_i)}\big). \tag{4.13}$$

By the fact that $\pi_k w - \widehat{I}_{k-1}\pi_{k-1} w$ is a continuous, piecewise linear functions on T and by (4.1), (4.8) and (4.13), we have

$$\begin{aligned}
\|\pi_k w - \widehat{I}_{k-1}\pi_{k-1} w\|_{L^2(T)} &\le C h_k\Big(\sum_{i=1,2,3} |\pi_k w(X_i) - \widehat{I}_{k-1}\pi_{k-1} w(X_i)| + \sum_{i=1,2,3} |\pi_k w(m_i) - \widehat{I}_{k-1}\pi_{k-1} w(m_i)|\Big) \\
&= C h_k \sum_{i=1,2,3} |\pi_k w(m_i) - \widehat{I}_{k-1}\pi_{k-1} w(m_i)| \\
&\le C h_k^{1+\alpha}\Big(\|w\|_{H^{1+\alpha}(T)} + \sum_{i=1,2,3} \|w\|_{H^{1+\alpha}(T_i)}\Big).
\end{aligned}$$

By summing over $T \in \mathcal{T}_{k-1}$, we obtain the desired inequality for case $m = 0$. The case when $m = 1$ is obtained from the standard inverse inequality. □

4.2 Proof of Theorem 4.1

It suffices to show (A.2) and (A.3). Firstly, we need following lemma.

Lemma 4.7. *There exists a constant* $C > 0$ *such that*

$$\|\widehat{I}_k u\|_{L^2(\Omega)} \le C\|u\|_{L^2(\Omega)} \tag{4.14}$$

for $u \in V_{k-1}(\Omega)$.

Proof. When $T \in \mathcal{T}_{k-1}$ is a non-interface element in $\mathcal{T}_{h_k}$, then $\widehat{I}_k u|_T = u|_T$. Thus, $(u, u)_T = (\widehat{I}_k u, \widehat{I}_k u)_T$. Assume T is an interface element. Suppose X_1, X_2 and X_3 are nodes of triangle T, and m_1, m_2 and m_3 are mid points of X_i $(i = 1, 2, 3)$ (see Figure 6). Then, by the definition of $\widehat{I}_k$, we have $u(X_i) = \widehat{I}_k u(X_i)$, $i = 1, 2, 3$. Also, the values of $\widehat{I}_k u$ at m_i are intermediate values of $u(X_i)$ and $u(X_{i+1})$ (here, $X_4 = X_1$). This is how $\widehat{I}_k u$ is constructed. In fact, because the diffusion coefficient β does not change sign across the interface, the values on the edges are bounded by the values on the nodes. See details in [10]. By (4.1), we have,

$$\|\widehat{I}_k u\|_{0,T} \le Ch_k \sum_{i=1}^{3} |\widehat{I}_k u(X_i)| + Ch_k \sum_{i=1}^{3} |\widehat{I}_k u(m_i)| \le Ch_k \sum_{i=1}^{3} |u(X_i)| \le C\|u\|_{0,T}.$$

By summing over all elements $T \in \mathcal{T}_k$, we have the conclusion. □

Lemma 4.8. *There exists a constant* $C > 0$ *such that*

$$\|\widehat{I}_k u\|_{1,h} \le C\|u\|_{1,h} \tag{4.15}$$

for all $u \in V_{k-1}(\Omega)$.

Proof. When $T \in \mathcal{T}_{k-1}$ is a non-interface element, then $\widehat{I}_k u|_T = u|_T$. Assume T is an interface element. We refer to Figure 6. By the definition of $\widehat{I}_k$, we have $u(X_i) = \widehat{I}_k u(X_i)$, $i = 1, 2, 3$. Let $a = u(X_1)$. Since $u - a$ and $\widehat{I}_k u - a$ is an H^1-function on T vanishing at X_1, there exists some constant $C > 0$ (see [11]) such that

$$\frac{1}{C} \max_T |\widehat{I}_k u - a| \le |\widehat{I}_k u|_{1,T} \le C \max_T |\widehat{I}_k u - a|, \tag{4.16}$$

$$\frac{1}{C} \max_T |u - a| \le |u|_{1,T} \le C \max_T |u - a|. \tag{4.17}$$

By (4.16), we have

$$|\widehat{I}_k u|_{1,T} \le C \max_{i=1,2,3} \{|\widehat{I}_k u(X_i) - a|, |\widehat{I}_k u(m_i) - a|\}. \tag{4.18}$$

By the fact that the values of $\widehat{I}_k u$ at m_i are intermediate values of $u(X_i)$ and $u(X_{i+1})$ and by (4.17), we have

$$\max_{i=1,2,3} \{|\widehat{I}_k u(X_i) - a|, |\widehat{I}_k u(m_i) - a|\} \le C \max_{i=1,2,3} |u(X_i) - a| \le C|u|_{1,T}. \tag{4.19}$$

Combining (4.18), (4.19), and (4.14), we have the desired inequality. □

We now prove (A.2).

Theorem 4.9. *There exists a constant* $C^* > 0$ *such that, for all* $u \in V_{k-1}(\Omega)$ *the following holds:*

$$|||\widehat{I}_k u|||_k \le C^* |||u|||_{k-1}. \tag{4.20}$$

Proof. The desired inequality follows directly by (2.2) and (4.15). □

Corollary 4.10. *For all* $u_k \in V_k(\Omega)$ *the following holds:*

$$|||P_{k-1} u_k|||_{k-1} \le C^* |||u_k|||_k, \tag{4.21}$$

where the constat $C^* > 0$ *is same as in* (4.20).

Proof. By the Cauchy–Schwarz inequality and (4.20),

$$|||P_{k-1} u|||_{k-1}^2 = a_{k-1}(P_{k-1} u, P_{k-1} u) = a_k(u, \widehat{I}_k P_{k-1} u) \le |||u|||_k |||\widehat{I}_k P_{k-1} u|||_k \le C^* |||u|||_k |||P_{k-1} u|||_{k-1}.$$ □

Lemma 4.11. *There exists a constant $C > 0$ such that for all $u \in V_{k-1}(\Omega)$, the following holds:*

$$\|u - \widehat{I}_k u\|_{L^2(\Omega)} \le Ch_k |||u|||_{k-1}. \tag{4.22}$$

Proof. Let $\phi_k = u - \widehat{I}_k u$. When $T \in \mathcal{T}_{k-1}$ is a non-interface element, then $\|\phi_k\|_{0,T} = 0$. Suppose $T \in \mathcal{T}_{k-1}$ is an interface element with nodes X_1, X_2 and X_3 (see Figure 6). By the fact that $\phi_k(X_i) = 0$, $i = 1, 2, 3$, we have by the Poincaré inequality,

$$\|\phi\|_{0,T} \le Ch_k \|\phi\|_{1,T}.$$

The above inequality, the triangle inequality and (4.15) yield

$$\|\phi\|_{L^2(\Omega)} \le Ch_k \|\phi\|_{1,h_k} \le Ch_k(\|u\|_{1,h_k} + \|\widehat{I}_k u\|_{1,h_k}) \le Ch_k \|u\|_{1,h_k}.$$ ☐

Lemma 4.12. *For all $u \in V_k$,*

$$\|A_k u\|_{H^{-1}(\Omega)} \le C|||u|||_k. \tag{4.23}$$

Proof. We see that for any w in $H^1(\Omega)$ the following holds:

$$\begin{aligned} \frac{|(A_k u, w)|}{\|w\|_{H^1(\Omega)}} &\le \frac{|(A_k u, w - \pi_k w)|}{\|w\|_{H^1(\Omega)}} + \frac{|(A_k u, \pi_k w)|}{\|w\|_{H^1(\Omega)}} \\ &\le \frac{\|A_k u\|_{L^2(\Omega)} \|w - \pi_k w\|_{L^2(\Omega)}}{\|w\|_{H^1(\Omega)}} + \frac{|||u|||_k |||\pi_k w|||_k}{\|w\|_{H^1(\Omega)}} \\ &\le Ch_k \|A_k u\|_{L^2(\Omega)} + C|||u|||_k \le C|||u|||_k, \end{aligned}$$

where we used the interpolation properties (2.4) and (2.5). By taking supremum over $w \in H^1_0(\Omega)$, we have the desired inequality. ☐

Finally, we show that assumption (A.3) holds with $\nu = \frac{\alpha}{2}$.

Theorem 4.13. *There exists a number $0 < \nu < 1$ and a constant $C > 0$ such that for all $u \in V_k(\Omega)$, the following holds:*

$$|a_k((I - \widehat{I}_k P_{k-1})u, u)| \le C\Big(\frac{\|A_k u\|_0^2}{\lambda_k}\Big)^{\nu} a_k(u, u)^{1-\nu}.$$

Proof. Consider the following dual problem: given $A_k u \in L^2(\Omega)$,

$$\begin{cases} -\nabla \cdot (\beta \nabla w) = A_k u & \text{in } \Omega, \\ [w]_\Gamma = \Big[\beta \dfrac{\partial w}{\partial \mathbf{n}}\Big]_\Gamma = 0, \\ w = 0 & \text{on } \partial\Omega. \end{cases}$$

Note that there exists a solution w in $H^2(\Omega^-) \cap H^2(\Omega^+) \cap H^{1+\alpha}_{0,\Gamma}(\Omega)$ such that

$$\|w\|_{H^{1+\alpha}(\Omega)} \le C\|A_k u\|_{H^{-1+\alpha}(\Omega)}. \tag{4.24}$$

By the definition of $a_k(\cdot, \cdot)$ and A_k, we see that u is an elliptic projection of w onto $V_k(\Omega)$, i.e.,

$$a_k(w, \phi_k) = (A_k u, \phi_k)_k, \quad a_k(u, \phi_k) = (A_k u, \phi_k)_k \quad \text{for all } \phi_k \in \widehat{S}_k(\Omega).$$

Hence, we have

$$|||u - w|||_k \le Ch_k^\alpha \|w\|_{H^{1+\alpha}(\Omega)}. \tag{4.25}$$

Using the triangle inequality and (4.25), (2.4) and (4.24), we obtain

$$|||u - \pi_k w|||_k \le |||u - w|||_k + |||w - \pi_k w|||_k \le Ch_k^\alpha \|w\|_{H^{1+\alpha}(\Omega)} \le Ch_k^\alpha \|A_k u\|_{H^{-1+\alpha}(\Omega)}. \tag{4.26}$$

By the definition of $\widehat{I}_k$,

$$\begin{aligned} a_k((I - \widehat{I}_k P_{k-1})u, u) &= a_k(u, u) - a_{k-1}(P_{k-1}u, P_{k-1}u) \\ &= a_k(u - \pi_k w, u) + a_{k-1}(\pi_{k-1}w - P_{k-1}u, P_{k-1}u) + a_k(\pi_k w, u) - a_{k-1}(\pi_{k-1}w, P_{k-1}u) \\ &=: \Phi_1 + \Phi_2 + \Phi_3. \end{aligned}$$

Using the Cauchy–Schwarz inequality, (4.26), interpolation between spaces [8] and (4.23), we get

$$\begin{aligned}|\Phi_1| &\le Ch_k^{\alpha}\|A_k u\|_{H^{-1+\alpha}(\Omega)}|||u|||_k\\ &\le Ch_k^{\alpha}\|A_k u\|_{H^{-1}(\Omega)}^{1-\alpha}\|A_k u\|_{L^2(\Omega)}^{\alpha}|||u|||_k\\ &\le Ch_k^{\alpha}|||u|||_k^{1-\alpha}\|A_k u\|_{L^2(\Omega)}^{\alpha}|||u|||_k = Ch_k^{\alpha}|||u|||_k^{2-\alpha}\|A_k u\|_{L^2(\Omega)}^{\alpha}.\end{aligned} \tag{4.27}$$

By the definition of $\widehat{I}_k$ and by the Cauchy–Schwarz inequality, (2.2) and (4.7),

$$\Phi_3 = a_k(\pi_k w - \widehat{I}_k\pi_{k-1}w, u) \le Ch_k^{\alpha}|||w|||_{H^{1+\alpha}(\Omega)}|||u|||_k.$$

Similar techniques as above yield

$$|\Phi_3| \le Ch_k^{\alpha}|||u|||_k^{2-\alpha}\|A_k u\|_{L^2(\Omega)}^{\alpha}. \tag{4.28}$$

To bound Φ_2, we define an operator $\widetilde{P}_{k-1} : V_k(\Omega) \to V_{k-1}(\Omega)$ defined by

$$a_{k-1}(u, \widetilde{P}_{k-1}, v) = a_k(u, v) \quad \text{for all } u \in V_k(\Omega),\ v \in V_{k-1}(\Omega).$$

By the definition of the operator $\widetilde{P}_{k-1}$ and P_{k-1}, we rewrite Φ_2 as

$$\begin{aligned}\Phi_2 &= a_{k-1}(\pi_{k-1}w - \widetilde{P}_{k-1}u, P_{k-1}u) + a_{k-1}(\widetilde{P}_{k-1}u - P_{k-1}u, P_{k-1}u)\\ &= a_{k-1}(\pi_{k-1}w - \widetilde{P}_{k-1}u, P_{k-1}u) + a_k(u, (I - \widehat{I}_k)P_{k-1}u) =: \Phi_{2_a} + \Phi_{2_b}.\end{aligned} \tag{4.29}$$

Using the similar technique as (4.26), we have

$$\begin{aligned}|\Phi_{2_a}| &= |a_{k-1}(\pi_{k-1}w, P_{k-1}u) - a_k(u, P_{k-1}u)|\\ &\le |a_{k-1}(\pi_{k-1}w - w, P_{k-1}u)| + |a_k(u - w, P_{k-1}u)|\\ &\le Ch_k^{\alpha}|||u|||_k^{2-\alpha}\|A_k u\|_{L^2(\Omega)}^{\alpha}.\end{aligned} \tag{4.30}$$

By the definition of A_{k-1} and the Cauchy–Schwarz inequality, (4.22) and (4.21), we obtain

$$|\Phi_{2_b}| = |(A_k u, (I - \widehat{I}_k)P_{k-1}u))| \le \|A_k u\|_{L^2(\Omega)} \cdot Ch_k|||P_{k-1}u|||_{k-1} \le Ch_k\|A_k u\|_{L^2(\Omega)}|||u|||_k. \tag{4.31}$$

Due to (2.3),

$$\|A_k u\|_{L^2(\Omega)} = \|A_k u\|_{L^2(\Omega)}^{\alpha} \cdot \|A_k u\|_{L^2(\Omega)}^{1-\alpha} \le \|A_k u\|_{L^2(\Omega)}^{\alpha} \cdot Ch_k^{-1+\alpha}|||u|||_k^{1-\alpha}. \tag{4.32}$$

Hence, by (4.31) and (4.32) we have

$$|\Phi_{2_b}| \le Ch_k\|A_k u\|_{L^2(\Omega)}^{\alpha} \cdot Ch_k^{-1+\alpha}|||u|||_{k-1}^{1-\alpha}|||u|||_k = Ch_k^{\alpha}|||u|||_k^{2-\alpha}\|A_k u\|_{L^2(\Omega)}^{\alpha}. \tag{4.33}$$

Using (4.27), (4.28), (4.29), (4.30) and (4.33), we obtain

$$|a_k((I - \widehat{I}_k P_{k-1})u, u)| \le Ch_k^{\alpha}|||u|||_k^{2-\alpha}\|A_k u\|_{L^2(\Omega)}^{\alpha}.$$

Hence, by the definition of $|||\cdot|||_{k-1}$ and by (2.3), together with the fact that $\lambda_k = \mathcal{O}(h_k^{-2})$, we have

$$|a_k((I - \widehat{I}_k P_{k-1})u, u)| \le C\Big(\frac{\|A_k u\|_0^2}{\lambda_k}\Big)^{\frac{\alpha}{2}} a_k(u, u)^{1-\frac{\alpha}{2}}. \qquad \square$$

5 Numerical Results

In this section, we demonstrate the performance of our multigrid algorithm $\mathbf{SUMG}_J$. We report the number of $\mathcal{V}_q(m, m)$ iterations, and total CPU-time to reach the stopping criteria,

$$\frac{\|\widetilde{f}_J - \widetilde{A}_J x\|}{\|\widetilde{f}_J\|} < 10^{-6}.$$

We present two examples. For the first example, we report the performance of $\mathbf{SUMG}_J$ in Table 1, where many different ratios of β are considered, i.e., $\frac{\beta^-}{\beta^+} = 1, 10, 100, 1000$. We see that the number of $\mathcal{V}_3(3,3)$ cycles increases as the ratio of β increases. However, the number of cycles remain bounded as level J increases. For the second example, we report the performance of $\mathbf{SUMG}_J$ with $\mathcal{V}_2(2,2)$ of the case $\frac{\beta^-}{\beta^+} = \frac{1}{100}$ in Table 2 where non-convex subdomain Ω^- is considered.

For both the examples, we compare the performance of $\mathbf{SUMG}_J$ with that of diagonally-preconditioned conjugate gradient methods (D-PCG).

The domain Ω is $[-1,1]^2$ for both the examples. The subdomain Ω^- is defined as $\{(x,y) \in \Omega : L(x,y) < 0\}$ for some level set function $L(x,y)$, and $\Omega^+ = \Omega/\Omega^-$. We use uniform hierarchical triangulations $\mathcal{T}_k$ with mesh size $h_k = 2^{-k}h_0$ $(k = 0, 1, \dots, J)$. A mesh $\mathcal{F}_J$ is obtained from the finest uniform mesh $\mathcal{T}_J$ by refining interface elements using the intersection points. We used computation environment of Intel(R) Core(TM) i7-3770 CPU @ 3.40GHz processor.

Example 1

The level function is $L(x,y) = x^2 + y^2 - r_0^2$, where $r_0 = 0.48$. The exact solution $u(x,y)$ is

$$u = \begin{cases} \frac{r^3}{\beta^-} & \text{in } \Omega^-, \\ \frac{r^3}{\beta^+} + \left(\frac{1}{\beta^-} - \frac{1}{\beta^+}\right) r_0^3 & \text{in } \Omega^+. \end{cases}$$

We used $\mathcal{V}_3(3,3)$ for the semi-uniform multigrid algorithm $\mathbf{SUMG}_J$. We report the performance of $\mathbf{SUMG}_J$ and D-PCG for the various cases of β jumps in Table 1. We see that the number of $\mathcal{V}_3(3,3)$ cycles increases as the ratio of β increases from 1 to 1000. This is natural since the condition number of $\widetilde{A}_J$ increases as $\frac{\beta^-}{\beta^+}$ increases. However, for the fixed ratio of $\frac{\beta^-}{\beta^+}$, the iteration numbers of $\mathcal{V}_3(3,3)$ remain uniformly bounded as level J increases. The CPU time of $\mathbf{SUMG}_J$ grows like $\mathcal{O}(N)$ while that of D-PCG grow like $\mathcal{O}(N^{\frac{3}{2}})$.

	Case 1						Case 2				
	$\mathcal{V}_3(3,3)$-cycle			D-PCG			$\mathcal{V}_3(3,3)$-cycle			D-PCG	
$\frac{1}{h_J}$	Iter.	δ	CPU time	Iter.	CPU time	$\frac{1}{h_J}$	Iter.	δ	CPU time	Iter.	CPU time
16	8	0.145	0.188	90	0.031	16	11	0.272	0.249	98	0.033
32	8	0.165	0.331	165	0.191	32	13	0.341	0.546	172	0.194
64	8	0.167	0.779	300	1.294	64	15	0.371	1.487	306	1.422
128	8	0.164	2.626	490	9.462	128	15	0.382	4.792	553	10.688
256	8	0.159	9.320	951	70.139	256	15	0.390	18.025	984	75.693

	Case 3						Case 4				
	$\mathcal{V}_3(3,3)$-cycle			D-PCG			$\mathcal{V}_3(3,3)$-cycle			D-PCG	
$\frac{1}{h_J}$	Iter.	δ	CPU time	Iter.	CPU time	$\frac{1}{h_J}$	Iter.	δ	CPU time	Iter.	CPU time
16	14	0.371	0.320	109	0.040	16	17	0.443	0.395	121	0.043
32	18	0.460	0.756	190	0.205	32	23	0.510	0.963	205	0.228
64	21	0.514	2.084	345	1.487	64	27	0.595	2.551	382	1.661
128	21	0.517	6.620	630	12.191	128	33	0.655	9.894	710	13.492
256	23	0.546	27.649	1120	82.646	256	30	0.615	35.132	759	56.057

Table 1: The number of iterations, contraction number δ, and CPU time of $\mathbf{SUMG}_J$ and number of iterations and CPU time of D-PCG for Example 1 with various jumps of β. Case 1, Case 2, Case 3 and Case 4 correspond to $\frac{\beta^-}{\beta^+} = 1$, $\frac{\beta^-}{\beta^+} = 10$, $\frac{\beta^-}{\beta^+} = 100$ and $\frac{\beta^-}{\beta^+} = 1000$, respectively.

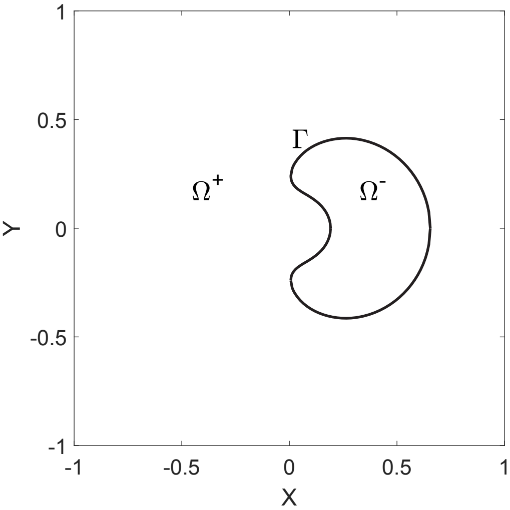

Figure 8: Interface and subdomains of Example 2.

	$\mathcal{V}_3(3,3)$-cycle			D-PCG	
$\frac{1}{h_J}$	Iter.	δ	CPU time	Iter.	CPU time
16	10	0.231	0.213	146	0.031
32	11	0.271	0.428	389	0.245
64	12	0.310	0.994	624	1.454
128	13	0.337	3.326	940	8.881
256	14	0.367	12.809	1721	71.014

Table 2: The number of iterations, contraction number δ, and CPU time of $\mathbf{SUMG}_J$ and number of iterations and CPU time of $\mathbf{SUMG}_J$ and D-PCG for Example 2.

Example 2

We consider an example whose subdomain Ω^- is non-convex. The level set function is

$$L(x,y) = (3x^2 + 3y^2 - x)^2 - (x^2 + y^2) + 0.03.$$

We refer to Figure 8. The exact solution $u(x,y)$ is

$$u = \begin{cases} \frac{x((3x^2+3y^2-x)^2-(x^2+y^2)+0.03)}{\beta^-} & \text{in } \Omega^-, \\ \frac{x((3x^2+3y^2-x)^2-(x^2+y^2)+0.03)}{\beta^+} & \text{in } \Omega^+. \end{cases}$$

We report the performance of $\mathbf{SUMG}_J$ with $\mathcal{V}_2(2,2)$ when $\beta^- = 1, \beta^+ = 100$ in Table 2. We see that the numbers of cycles of $\mathcal{V}_2(2,2)$ remain bounded as J increases. The computational complexity of $\mathbf{SUMG}_J$ is $\mathcal{O}(N)$ while that of D-PCG is $\mathcal{O}(N^{\frac{3}{2}})$.

6 Conclusion

In this work, we proposed a semi-uniform multigrid algorithm (SUMG) for elliptic interface problems. We use P_1-conforming method on a semi-uniform grid for the discretization of the problems where a semi-uniform grid is obtained by refining uniform grid at interface points. We adopt subspace correction methods where we choose uniform grids as the auxiliary space. The transfer operator is defined so that the transferred functions on a semi-uniform grid satisfy the flux continuity across the interface. On the auxiliary space, we use multigrid algorithm where the prolongation operators are modified.

We prove the contracting property of the proposed multigrid algorithm. We test SUMG for elliptic interface problems where different β ratios are considered. We see that as the β ratio increases, the number of $\mathcal{V}$-cycles of SUMG increases. However, for fixed β ratio we observe that the number of $\mathcal{V}$-cycles of SUMG remain uniformly bounded as $h \to 0$. We also compared SUMG with D-PCG. We observe that the computational complexity of SUMG is $\mathcal{O}(N)$ for all problems while that of D-PCG is $\mathcal{O}(N^{3/2})$.

Funding: This work is supported by the National Research Foundation of Korea (NRF) grant funded by the Korea government (MSIT) (No. 2020R1C1C1A01005396).

References

[1] L. Adams and T. P. Chartier, New geometric immersed interface multigrid solvers, *SIAM J. Sci. Comput.* **25** (2004), no. 5, 1516–1533.

[2] L. Adams and Z. Li, The immersed interface/multigrid methods for interface problems, *SIAM J. Sci. Comput.* **24** (2002), no. 2, 463–479.

[3] I. Babuška, The finite element method for elliptic equations with discontinuous coefficients, *Computing* **5** (1970), 207–213.

[4] J. H. Bramble and J. T. King, A finite element method for interface problems in domains with smooth boundaries and interfaces, *Adv. Comput. Math.* **6** (1996), no. 2, 109–138.

[5] J. H. Bramble and J. E. Pasciak, New convergence estimates for multigrid algorithms, *Math. Comp.* **49** (1987), no. 180, 311–329.

[6] J. H. Bramble and J. E. Pasciak, The analysis of smoothers for multigrid algorithms, *Math. Comp.* **58** (1992), no. 198, 467–488.

[7] J. H. Bramble, J. E. Pasciak and J. Xu, The analysis of multigrid algorithms with nonnested spaces or noninherited quadratic forms, *Math. Comp.* **56** (1991), no. 193, 1–34.

[8] A.-P. Calderón, Intermediate spaces and interpolation, the complex method, *Studia Math.* **24** (1964), 113–190.

[9] L. Chen, H. Wei and M. Wen, An interface-fitted mesh generator and virtual element methods for elliptic interface problems, *J. Comput. Phys.* **334** (2017), 327–348.

[10] S.-H. Chou, D. Y. Kwak and K. T. Wee, Optimal convergence analysis of an immersed interface finite element method, *Adv. Comput. Math.* **33** (2010), no. 2, 149–168.

[11] P. G. Ciarlet, *The Finite Element Method for Elliptic Problems*, Class. Appl. Math. 40, Society for Industrial and Applied Mathematics, Philadelphia, 2002.

[12] R. D. Falgout, An introduction to algebraic multigrid computing, *Comput. Sci. Eng.* **8** (2006), 24–33.

[13] W. Hackbusch, *Multigrid Methods and Applications*, Springer Ser. Comput. Math. 4, Springer, Berlin, 1985.

[14] R. Hiptmair and J. Xu, Nodal auxiliary space preconditioning in $\mathbf{H}(\mathbf{c}url)$ and $\mathbf{H}(\mathrm{div})$ spaces, *SIAM J. Numer. Anal.* **45** (2007), no. 6, 2483–2509.

[15] X. Hu, W. Liu, G. Qin, J. Xu and Z. Zhang, Development of a fast auxiliary subspace pre-conditioner for numerical reservoir simulators, in: *SPE Reservoir Characterisation and Simulation Conference and Exhibition*, Society of Petroleum Engineers, Texas (2011), SPE-148388-MS.

[16] J. Huang and J. Zou, Some new a priori estimates for second-order elliptic and parabolic interface problems, *J. Differential Equations* **184** (2002), no. 2, 570–586.

[17] G. Jo and D. Y. Kwak, An IMPES scheme for a two-phase flow in heterogeneous porous media using a structured grid, *Comput. Methods Appl. Mech. Engrg.* **317** (2017), 684–701.

[18] G. Jo and D. Y. Kwak, Geometric multigrid algorithms for elliptic interface problems using structured grids, *Numer. Algorithms* **81** (2019), no. 1, 211–235.

[19] D. Y. Kwak and J. Lee, A modified P_1-immersed finite element method, *Int. J. Pure Appl. Math.* **104** (2015), 471–494.

[20] D. Y. Kwak, K. T. Wee and K. S. Chang, An analysis of a broken P_1-nonconforming finite element method for interface problems, *SIAM J. Numer. Anal.* **48** (2010), no. 6, 2117–2134.

[21] C. O. Lee, A nonconforming multigrid method using conforming subspaces, in: *Proceedings of the Sixth Copper Mountain Conference on Multigrid Methods*, NASA, Hampton (1993), 317–330.

[22] Z. Li, T. Lin, Y. Lin and R. C. Rogers, An immersed finite element space and its approximation capability, *Numer. Methods Partial Differential Equations* **20** (2004), no. 3, 338–367.

[23] Z. Li, T. Lin and X. Wu, New Cartesian grid methods for interface problems using the finite element formulation, *Numer. Math.* **96** (2003), no. 1, 61–98.

[24] S. F. McCormick, *Multigrid Methods*, Front. Appl. Math. 3, Society for Industrial and Applied Mathematics, Philadelphia, 1987.

[25] J. A. Roĭtberg and Z. Šeftel’, A theorem on homeomorphisms for elliptic systems and its applications, *Sb. Math.* **7** (1969), 439–465.
[26] K. Stüben, A review of algebraic multigrid, *J. Comput. Appl. Math.* **1–2** (2001), 281–309.
[27] P. Vaněk, M. Brezina and J. Mandel, Convergence of algebraic multigrid based on smoothed aggregation, *Numer. Math.* **88** (2001), no. 3, 559–579.
[28] J. Xu, The auxiliary space method and optimal multigrid preconditioning techniques for unstructured grids, *Computing* **56** (1996), 215–235.